Journal of Applied Mechanics and Technical Physics, Vol. 50, No. 4, pp. 698–700, 2009

ON DETERMINING STRESSES IN RIGID INCLUSIONS. PLANE PROBLEMS

I. Yu. Tsvelodub

UDC 539.3

Plane problems of determining the stress–strain state of an isotropic elastic domain with a rigid inclusion are considered. It is shown that the stress field in the inclusion is uniquely determined. This field is uniform for a plane with an elliptic inclusion, and the stresses at infinity and in the inclusion are related by mutually single-valued formulas.

Key words: *plane elastic domain, rigid inclusion, uniform fields of stresses and moments.*

The model of an absolutely rigid body is an idealized presentation of the behavior of a real material: the medium starts to deform only when a certain level of stresses is reached. For instance, this approach is used in the theory of a rigid-plastic medium [1], where there are no deformations at $s < \sigma_{\rm yield}$ (s is a certain invariant of the stress tensor, for instance, the intensity of shear stresses and $\sigma_{\rm yield}$ is the yield stress), the material starts to experience plastic deformation at $s = \sigma_{\rm yield}$, and material failure may occur under further intensification of loading. Thus, information about the stress state in the rigid inclusion (RI) can be useful for estimating its strength.

The problem of determining the stress–strain state of an elastic plane with a physically nonlinear elliptic inclusion (PNEI) under conditions of plane deformation or in the generalized plane stress state was considered in [2, 3]. Relations between the uniform stress fields at infinity and in the inclusion were established. Similar results were obtained in [4] for the problem of pure bending of an infinite elastic plate containing a PNEI. Relations obtained in [2–4] are also applicable in the case of a rigid inclusion. Moreover, a plane problem of determining stresses in a finite elastic domain with an arbitrarily shaped RI is well-posed if the inclusion is assumed to be elastic and to have an infinitely large shear modulus.

1. Formulation of the Problem. Let us consider an isotropic elastic domain S with a rigid inclusion S^* under conditions of plane deformation or in the generalized plane stress state. External boundaries of the domain with the RI are simple closed contours L and L^* (L^* is the boundary between the domain S and the RI S^*). The domain S obeys Hooke's law [2]

$$8\mu\varepsilon_{kl} = (\varkappa - 1)\sigma_{nn}\delta_{kl} + 4\sigma^0_{kl},$$
$$\sigma^0_{kl} = \sigma_{kl} - \sigma_{nn}\delta_{kl}/2 \qquad (k, l = 1, 2), \tag{1.1}$$

where σ^0_{kl} and δ_{kl} are the components of the plane stress deviator and the unit tensor, μ is the shear modulus, $\varkappa = 3 - 4\nu$ in the case of plane deformation and $\varkappa = (3 - \nu)/(1 + \nu)$ in the case of the generalized plane stress state, and ν is Poisson's ratio; summation is performed over repeated indices from 1 to 2.

The strains ε_{kl} are assumed to be small and to be expressed in terms of the displacements u_k $(k, l = 1, 2)$ by the known Cauchy relations.

For the RI, i.e., for the domain S^*, we obtain

$$\varepsilon^*_{kl} = 0, \qquad u^*_{kl} = 0 \qquad (k, l = 1, 2). \tag{1.2}$$

Lavrent'ev Institute of Hydrodynamics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090; itsvel@hydro.nsc.ru. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 50, No. 4, pp. 183–186, July–August, 2009. Original article submitted April 23, 2008.

The displacements u_k or the loads $p_k = \sigma_{kl} n_l$ are specified on the boundary L (n_k are the components of the unit vector of the normal to L). The displacements and loads on the boundary L^* between the domains S^* and S are continuous; therefore, $u_k = 0$ on L^* by virtue of Eq. (1.2). Thus, for the domain S, we obtain a problem in displacements or a mixed problem whose solution gives us the stress field σ_{kl} and, consequently, the loads $p_k^* = p_k \equiv \sigma_{kl} n_l^*$ on L^* (n_k^* are the components of the unit vector of the normal to L^*).

Considering further the RI S^* as an elastic medium with the shear modulus μ^* ($\mu^* \to \infty$), we obtain a biharmonic equation for the stress function F^* with the quantities F^* and $\partial F^*/\partial n^*$ specified on L^*. This problem has been well studied [5], and its solution (which is unique) yields the stress field σ_{kl}^* in the domain S^*.

2. Elastic Plane with an Elliptic RI. Let us consider a particular case of the above-formulated problem. The domain S is an elastic plane with the behavior of the material in this domain being described by Hooke's law (1.1), and the contour L^* is an ellipse described in the coordinate system Oxy by the formula $x^2 a^{-2} + y^2 b^{-2} = 1$, $a \geq b$. Therefore, S^* is an elliptic rigid inclusion (ERI). Uniformly distributed stresses σ_{kl}^∞ are applied at infinity, and there is no rotation.

A similar problem was considered in [2, 3] for the PNEI with the constitutive relations of a rather general form:

$$\varepsilon_{kl}^* = F_{kl}(\sigma_{mn}^*) \qquad (k, l, m, n = 1, 2) \tag{2.1}$$

(F_{kl} are, generally speaking, nonlinear operators that describe, for instance, the elastoviscoplastic properties of the medium). It is shown that the stress field in the inclusion is uniform, and the following relations are established between the stress–strain states in the PNEI and at infinity:

$$\begin{gathered}
F_i = \alpha_{ij} y_j + \beta_{ij} x_j \quad (i = 1, 2, 3), \qquad F_1 = \varepsilon_{11}^*, \quad F_2 = \varepsilon_{22}^*, \quad F_3 = 2\varepsilon_{12}^*, \\
y_1 = \sigma_{11}^*, \quad y_2 = \sigma_{22}^*, \quad y_3 = \sigma_{12}^*, \qquad x_1 = \sigma_{11}^\infty, \quad x_2 = \sigma_{22}^\infty, \quad x_3 = \sigma_{12}^\infty, \\
\alpha_{11} = -\frac{(\varkappa + 1)(1 - m)}{4\mu(1 + m)}, \qquad \alpha_{12} = \alpha_{21} = \frac{\varkappa - 1}{4\mu}, \qquad \alpha_{22} = -\frac{(\varkappa + 1)(1 + m)}{4\mu(1 - m)}, \\
\alpha_{33} = -\frac{\varkappa + m^2}{\mu(1 - m^2)}, \qquad \beta_{11} = \frac{(\varkappa + 1)(3 - m)}{8\mu(1 + m)}, \qquad \beta_{12} = \beta_{21} = -\frac{\varkappa + 1}{8\mu}, \\
\beta_{22} = \frac{(\varkappa + 1)(3 + m)}{8\mu(1 - m)}, \qquad \beta_{33} = \frac{\varkappa + 1}{\mu(1 - m^2)}, \qquad m = \frac{a - b}{a + b} \quad (0 \leq m < 1);
\end{gathered} \tag{2.2}$$

the remaining coefficients α_{ij} and β_{ij} are equal to zero. In Eqs. (2.2), summation is performed over j from 1 to 3.

Relations (2.1) and (2.2) form a closed system for determining σ_{kl}^* on the basis of the stresses at infinity σ_{kl}^∞. This system is uniquely resolved with respect to σ_{kl}^* under the conditions indicated in [2, 3]; vice versa, using Eqs. (2.1) and (2.2), one can find the stresses σ_{kl}^∞ that ensure the necessary stresses σ_{kl}^* in S^*. The field σ_{kl}^∞ is determined uniquely, because the matrix $\|\beta_{ij}\|$ in Eqs. (2.2) is positively determined.

In the case considered, the rigid inclusion can be considered as a PNEI whose material is described by the constitutive equations of the form (1.2). Then, Eqs. (2.2) yield the system

$$\alpha_{ij} y_j + \beta_{ij} x_j = 0 \qquad (i = 1, 2, 3), \tag{2.3}$$

which uniquely determines y_i (i.e., the stresses σ_{kl}^* in the inclusion), because the matrix $\|\alpha_{ij}\|$ is negatively determined. This solution has the form

$$\begin{gathered}
\sigma_{11}^* = \frac{\varkappa + 1}{4\varkappa(1 - m)} [(\varkappa + 2 - m)\sigma_{11}^\infty + (\varkappa - 2 - m)\sigma_{22}^\infty], \\
\sigma_{22}^* = \frac{\varkappa + 1}{4\varkappa(1 + m)} [(\varkappa - 2 + m)\sigma_{11}^\infty + (\varkappa + 2 + m)\sigma_{22}^\infty], \qquad \sigma_{12}^* = \frac{\varkappa + 1}{\varkappa + m^2} \sigma_{12}^\infty.
\end{gathered} \tag{2.4}$$

For instance, under uniaxial tension or compression at infinity along the first axis ($\sigma_{11}^\infty \neq 0$ and $\sigma_{22}^\infty = \sigma_{12}^\infty = 0$), Eqs. (2.4) yield $\operatorname{sign} \sigma_{11}^* = \operatorname{sign} \sigma_{11}^\infty$ and $\operatorname{sign} \sigma_{22}^* = \pm \operatorname{sign} \sigma_{11}^\infty$ depending on $\operatorname{sign}(\varkappa - 2 + m)$; for $\sigma_{22}^\infty \neq 0$ and $\sigma_{11}^\infty = \sigma_{12}^\infty = 0$, we obtain $\operatorname{sign} \sigma_{11}^* = \pm \operatorname{sign} \sigma_{22}^\infty$ depending on $\operatorname{sign}(\varkappa - 2 - m)$ and $\operatorname{sign} \sigma_{22}^* = \operatorname{sign} \sigma_{22}^\infty$; for $\sigma_{11}^\infty = \sigma_{22}^\infty$, we have $\operatorname{sign} \sigma_{11}^* = \operatorname{sign} \sigma_{22}^* = \operatorname{sign} \sigma_{11}^\infty$.

It follows from Eq. (2.4) that the shear and normal stresses at infinity affect only the corresponding stresses in the ERI: σ_{12}^{∞} affects σ_{12}^{*}; σ_{11}^{∞} and σ_{22}^{∞} affect σ_{11}^{*} and σ_{22}^{*}. Note also that Eqs. (2.4) yield $|\sigma_{11}^{\infty}| \to \infty$ as $m \to 1$ (thin rigid inclusion) for arbitrary stresses σ_{11}^{∞} and σ_{22}^{∞} if at least one of them differs from zero; the quantity σ_{22}^{*} is finite.

As was noted above, system (2.3) is uniquely resolved with respect to σ_{kl}^{∞}, i.e., for obtaining a given stress state in the ERI, it is necessary to create a particular stress field at infinity.

3. Pure Bending of a Plate with an ERI. Let us consider an infinite isotropic elastic plate of constant thickness h containing an ERI (of the same thickness h). The plate is subjected to pure bending under the action of uniformly distributed moments M_{kl}^{∞} at infinity. There are no surface loads. We have to determine the stress–strain state of the plate, in particular, the field of moments in the ERI.

A similar problem for the case of a PNEI with the constitutive equations of the form (2.1) resolved with respect to σ_{kl}^{*} was solved in [4]. It was shown that the field of moments M_{kl} and the corresponding curvatures $\varkappa_{kl}$ in the inclusion is uniform. In addition, the following relations were obtained:

$$D(1-\nu)\varkappa_{11} = a_{11}M_{11} + a_{12}M_{22} + b_{11}M_{11}^{\infty} + b_{12}M_{22}^{\infty},$$

$$D(1-\nu)\varkappa_{22} = a_{12}M_{11} + a_{22}M_{22} + b_{12}M_{11}^{\infty} + b_{22}M_{22}^{\infty},$$

$$D(1-\nu)\varkappa_{12} = a_0M_{12} + b_0M_{12}^{\infty}, \qquad a_{11} = f_1(m), \qquad a_{22} = f_1(-m),$$

$$a_{12} = -(1+\nu)/(3+\nu), \qquad a_0 = -(1-\nu)(1-m^2)/[4-(1-\nu)(1-m^2)], \tag{3.1}$$

$$b_{11} = f_2(m), \qquad b_{22} = f_2(-m), \qquad b_{12} = (1-\nu)/[(3+\nu)(1+\nu)], \qquad b_0 = 1-a_0,$$

$$f_1(x) = -2(1-x)/[(3+\nu)(1+x)], \quad f_2(x) = [3\nu+5+x(1-\nu)]/[(3+\nu)(1+\nu)(1+x)],$$

$$D = Eh^3/[12(1-\nu^2)].$$

Here D is the cylindrical rigidity, E is Young's modulus, and ν is Poisson's ratio of the elastic plate; the value of m is determined by Eqs. (2.2).

For the case of the rigid inclusion, assuming that $\varkappa_{kl} = 0$ and resolving system (3.1) with respect to M_{kl}, we find the expressions for the moments in the ERI:

$$(1-\nu^2)M_{11} = [\nu+\alpha(m)]M_{11}^{\infty} - [\nu\alpha(m)+1]M_{22}^{\infty},$$

$$(1-\nu^2)M_{22} = -[\nu\alpha(-m)+1]M_{11}^{\infty} + [\nu+\alpha(-m)]M_{22}^{\infty},$$

$$(1-\nu^2)M_{12} = 4(1+\nu)M_{12}^{\infty}/(1-m^2), \qquad \alpha(x) = (3+x)/(1-x).$$

Note that system (3.1) is uniquely resolved with respect to M_{kl}^{∞} as well, i.e., to obtain a uniform moment field M_{kl} in the ERI, it is necessary to ensure a uniform distribution of the above-indicated moments M_{kl}^{∞} at infinity.

This work was supported by the Russian Foundation for Basic Research (Grant No. 08-01-00168) and by the Council on the Grants of the President of the Russian Federation on Supporting the Leading Scientific Schools (Grant No. NSh-3066.2008.1).

REFERENCES

1. G. I. Bykovtsev and D. D. Ivlev, *Theory of Plasticity* [in Russian], Dal'nauka, Vladivostok (1998).
2. I. Yu. Tsvelodub, "Determination of the strength characteristics of a physically nonlinear inclusion in a linearly elastic medium," *J. Appl. Mech. Tech. Phys.*, **41**, No. 4, 734–739 (2000).
3. I. Yu. Tsvelodub, "Physically nonlinear inclusion in a linearly elastic medium (plane problem)," *Izv. Ross. Akad. Nauk, Mekh. Tverd. Tela*, No. 5, 72–84 (2000).
4. I. Yu. Tsvelodub, "Bending of elastic plates with a physically nonlinear inclusion," *J. Appl. Mech. Tech. Phys.*, **47**, No. 6, 899–904 (2006).
5. N. I. Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity*, Noordhoff, Groningen, Holland (1963).